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Dirac equations for MHD waves: Hamiltonian spectra and supersymmetry

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Abstract. The propagation of MHD waves in a one-dimensional atmosphere is considered. For Alfvén and acoustic waves the evolution equations are transformed into one-dimensional Dirac equations with a supersymmetric structure. For the case of isothermal and hydrostatic atmosphere rigorous spectral resolutions for the MHD Dirac Hamiltonians are given.

1. Introduction

Studies of the propagation of MHD waves in the solar and stellar atmospheres is important for the understanding of the energy transfer and dissipation in the chromosphere and corona [1]. In [2] the second-order wave equation for linear Alfvén waves has been transformed into the form of a one-dimensional Klein–Gordon equation with position dependent 'potential term'. We follow this idea and show that in the case of a vertical background magnetic field the first-order evolution equations for Alfvén and acoustic waves may be transformed into 1D Dirac equations with suitable potential terms. The obtained Dirac Hamiltonians display the structure of Witten's supersymmetric quantum mechanics [3,4]. Using the general results of the theory of Sturm–Liouville operators' [5,6] we argue that in a generic case the Dirac Hamiltonian for the Alfvén waves possesses discrete spectrum while for the acoustic waves the spectrum contains a continuous part. In the last section the general theory is applied to a simple model of an isothermal hydrostatic atmosphere in constant gravity and magnetic fields. For this model two theorems give a full rigorous spectral characterization of the Dirac Hamiltonians for Alfvén and acoustic waves. These results give a rigorous meaning to the existing analytical solutions [7,8], clarify the problem of boundary conditions and observed resonant behaviour for Alfvén waves. In a forthcoming publication [9] the coupling between Alfvén and acoustic waves will be considered using quantum mechanical perturbation calculus and the astrophysical consequences of the obtained numerical results will be discussed.

2. MHD waves in 1D model atmosphere

Consider a vertically stratified stationary plasma permeated by a stationary magnetic field. The equilibrium configuration of the plasma is described by the density $\rho_0(z)$,

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the pressure $p_0(z)$ and the background magnetic and gravitation fields denoted by $\boldsymbol{B} = (B_x(z), B_y(z), B_z(z))$ and $g = g(z)e_z$ respectively. At the moment we do not assume any additional conditions on ρ_0, p_0, $\boldsymbol{B}$, and g except for necessary smoothness. The variable z varies from 0 to $+\infty$.

Remark. The condition $\nabla \cdot \boldsymbol{B} = 0$ implies in our case that B_z = constant. However, the generalization to z-dependent B_z might be useful for future applications.

Starting from the basic MHD equations [1] one may easly derive the linear MHD equations for the z- and time-dependent perturbations $\rho(z;t)$, $p(z;t)$, $b_x(z;t)$, $b_y(z;t)$, $(b_z(z;t) = 0)$ of the density, pressure and magnetic field respectively and for the 'small' velocities $u_x(z;t)$, $u_y(z;t)$, $u_z(z;t)$.

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial z}(\rho_0 u_z) \tag{2.1}$$

$$\rho_0 \frac{\partial u_z}{\partial t} = -\frac{\partial p}{\partial z} - \frac{1}{\mu}\frac{\partial}{\partial z}(B_x b_x + B_y b_y) - g\rho \tag{2.2}$$

$$\rho_0 \frac{\partial u_x}{\partial t} = \frac{B_z}{\mu}\frac{\partial b_x}{\partial z} \tag{2.3}$$

$$\rho_0 \frac{\partial u_y}{\partial t} = \frac{B_z}{\mu}\frac{\partial b_y}{\partial z} \tag{2.4}$$

$$\frac{\partial b_x}{\partial t} = \frac{\partial}{\partial z}(B_z u_x) - \frac{\partial}{\partial z}(B_x u_z) \tag{2.5}$$

$$\frac{\partial b_y}{\partial t} = \frac{\partial}{\partial z}(B_z u_y) - \frac{\partial}{\partial z}(B_y u_z) \tag{2.6}$$

$$p = \Gamma(\rho, \boldsymbol{u}). \tag{2.7}$$

In the above equations μ is a constant magnetic permeability, (2.7) is a linear functional constraint equation which may be derived from the equation of state for the plasma and which enables one to eliminate p from the system (2.2)–(2.6).

3. Dirac equation for Alfvén waves

We assume now that $\boldsymbol{B}$ lies in the xz-plane, i.e.

$$B_y = 0. \tag{3.1}$$

In this case the fields u_y, b_y are decoupled from the others and form Alfvén waves satisfying the following equations:

$$\rho_0 \frac{\partial u_y}{\partial t} = \frac{B_z}{\mu}\frac{\partial b_y}{\partial z} \tag{3.2}$$

$$\frac{\partial b_y}{\partial t} = \frac{\partial}{\partial z}(B_z u_y). \tag{3.3}$$

We use transformations of fields and z similar to those in [2] and given by the formulae

$$\psi_1 = \sqrt{\nu/\mu} b_y \tag{3.4}$$

$$\psi_2 = -\sqrt{\rho_0 \nu} u_y \tag{3.5}$$

$$\xi = \int_0^z \nu^{-1}\,\mathrm{d}z' \qquad \xi \in [0, \xi_\infty) \qquad \xi_\infty = \int_0^\infty \nu^{-1}\,\mathrm{d}z \tag{3.6}$$

where

$$\nu(z) = \frac{v_\mathrm{A}(z)}{v_\mathrm{A}(0)} \tag{3.7}$$

and $v_\mathrm{A}(z) = B_z(z)/\sqrt{\mu\rho_0(z)}$ is the Alfvén velocity. Then the equations (3.2) and (3.3) are equivalent to

$$\frac{1}{c_\mathrm{A}}\frac{\partial\psi_1}{\partial t} = -\frac{\partial\psi_2}{\partial\xi} - m_\mathrm{A}\psi_2 \tag{3.8}$$

$$\frac{1}{c_\mathrm{A}}\frac{\partial\psi_2}{\partial t} = -\frac{\partial\psi_1}{\partial\xi} + m_\mathrm{A}\psi_1 \tag{3.9}$$

where $c_\mathrm{A} = v_\mathrm{A}(0)$ and

$$m_\mathrm{A} = m_\mathrm{A}(\xi) = \frac{1}{2}\frac{\partial}{\partial z}\nu(z). \tag{3.10}$$

Introducing the notation: $\sigma_k, k = 1, 2, 3, \ldots$ Pauli matrices $p_\mathrm{A} = -\mathrm{i}\frac{\partial}{\partial\xi}$, $\Psi = \binom{\psi_1}{\psi_2}$ we can write (3.8) and (3.9) as a 1D Dirac equation in the Hamiltonian form

$$\frac{\partial}{\partial t}\Psi = -\mathrm{i} H_\mathrm{A}\Psi \tag{3.11}$$

$$H_\mathrm{A} = c_\mathrm{A}(\sigma_1 p_\mathrm{A} + \sigma_2 m_\mathrm{A}). \tag{3.12}$$

The Hamiltonian H_A should be essentialy self-adjoint on the Hilbert space $\boldsymbol{H}_\mathrm{A} = L^2[0, \xi_\infty) \otimes C^2$. Then the Cauchy problem (3.11) is well posed and the solutions are given in terms of the one-parameter unitary group $\exp(-\mathrm{i}H_\mathrm{A}t)$ on $\boldsymbol{H}_\mathrm{A}$. Unitarity means here the preservation of the energy because

$$\begin{aligned}\|\Psi(t)\|^2 &= \int_0^{\xi_\infty} \mathrm{d}\xi\,[|\psi_1(\xi;t)|^2 + |\psi_2(\xi;t)|^2] \\ &= \int_0^\infty \mathrm{d}z\,[1/\mu|b_y|^2 + \rho_0|u_y|^2].\end{aligned} \tag{3.13}$$

In order to have a unique self-adjoint extention of H_A we need proper boundary conditions. For this idealized model there are no reasons to impose any particular boundary conditions at $\xi = \xi_\infty$ ($z = +\infty$), hence the end point ξ_∞ should be in the 'limit-point' case [5,6] which means that the proper boundary conditions at $\xi = \xi_\infty$ are automatically satisfied. Choosing other conditions we violate the self-adjointness of H_A and hence the conservation of energy. For $\xi = 0$ ($z = 0$) we have two boundary conditions which lead to a symmetric H_A and admit real solutions for (3.8) and (3.9)

$$\begin{aligned} &(a) \qquad \psi_1(0,t) = 0 \qquad \psi_2(0,t) \ \text{arbitrary} \\ &(b) \qquad \psi_2(0,t) = 0 \qquad \psi_1(0,t) \ \text{arbitrary}. \end{aligned} \tag{3.14}$$

In section 6 we shall illustrate these problems using an exactly solvable model.

One should notice that in the case of a vertical background magnetic field ($B_x = B_y = 0$) we obtain two modes of Alfvén waves $(u_x, b_x), (u_y, b_y)$ with orthogonal polarizations which are independent and decoupled from (ρ, u_z). One can describe them by a single Dirac equation starting with the complex valued fields $b = b_x + \mathrm{i}b_y, u = u_x + \mathrm{i}u_y$.

4. Dirac equation for acoustic waves

Assuming that the magnetic field $\boldsymbol{B}$ is vertical, i.e

$$B_x = B_y = 0 \tag{4.1}$$

we have an independent acoustic mode (ρ, u_z) satisfying the equations

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial z}(\rho_0 u_z) \tag{4.2}$$

$$\rho_0 \frac{\partial u_z}{\partial t} = -\frac{\partial p}{\partial z} - g\rho \tag{4.3}$$

$$p = \Gamma(\rho, u_z). \tag{4.4}$$

We impose now a local *isothermal condition* putting in place of (4.4)

$$p = \frac{p_0}{\rho_0}\rho = v_S^2 \rho \tag{4.5}$$

where $v_S(z) = \sqrt{p_0(z)/\rho_0(z)}$ is a local sound velocity for isothermal perturbations. The above condition is a quite reasonable approximation at least for small frequences [1].

The next assumption is a local static equilibrium condition

$$\frac{\mathrm{d}p_0}{\mathrm{d}z} = \frac{\mathrm{d}}{\mathrm{d}z}(v_S^2 \rho_0) = -g\rho_0. \tag{4.6}$$

Moreover we assume that

$$\left|\frac{\mathrm{d}}{\mathrm{d}z} \ln v_S\right| \ll \left|\frac{\mathrm{d}}{\mathrm{d}z} \ln \rho_0\right| \tag{4.7}$$

and hence we put the approximative simplifying condition

$$\frac{\mathrm{d}\rho_0}{\mathrm{d}z} = -\frac{g}{v_S^2}\rho_0. \tag{4.8}$$

Now we introduce rescaled fields and height variable given by the formulae

$$\phi_1 = \sqrt{\chi\rho_0}u_z \tag{4.9}$$

$$\phi_2 = \sqrt{\chi/\rho_0}\rho \tag{4.10}$$

$$\eta = \int_0^z \chi^{-1}\,\mathrm{d}z' \tag{4.11}$$

$$\chi(z) = \frac{v_S(z)}{v_S(0)}. \tag{4.12}$$

Then after simple calculations we obtain from (4.2) and (4.3) using the conditions (4.5) and (4.8) the Dirac-like equation for ϕ_1, ϕ_2

$$\frac{1}{c_S}\frac{\partial\phi_1}{\partial t} = -\frac{\partial\phi_2}{\partial\eta} - m_S\phi_2 \tag{4.13}$$

$$\frac{1}{c_S}\frac{\partial\phi_2}{\partial t} = -\frac{\partial\phi_1}{\partial\eta} + m_S\phi_1 \tag{4.14}$$

where $c_S = v_S(0)$ and

$$m_S = \frac{g}{c_S v_S}. \tag{4.15}$$

Again equations (4.13) and (4.14) can be transformed into the Dirac equation in the Hamiltonian form

$$\frac{\partial}{\partial t}\Phi = -\mathrm{i}H_S\Phi \tag{4.16}$$

with $\Phi = \binom{\phi_1}{\phi_2}, p_S = -\mathrm{i}\frac{\partial}{\partial\eta}$ and

$$H_S = c_S(\sigma_1 p_S + \sigma_2 m_S). \tag{4.17}$$

The boundary conditions at $\eta = 0$ $(z = 0)$ are the same as in the previous section and the second end point $\eta_\infty = \int_0^\infty \chi^{-1}\,\mathrm{d}z$ should be in the limit-point case too (typically $\eta_\infty = \infty$).

The conserved square of the norm is again the energy (per unit surface) given by the expression

$$\|\Phi(t)\|^2 = \int_0^{\eta_\infty} \mathrm{d}\eta\,[|\phi_1(\eta;t)|^2 + |\phi_2(\eta;t)|^2]$$
$$= \int_0^\infty \mathrm{d}z\,[\rho_0|u_z|^2 + p_0|\rho/\rho_0|^2]. \tag{4.18}$$

5. Supersymmetry and spectral properties

It is an amazing fact that the supersymmetry may be found in the astrophysical models. Namely, the Dirac operator $H_D = \sigma_1 p + \sigma_2 m$ which appears in the previous sections possesses properties of the supercharge operator in the supersymmetric quantum mechanics [3, 4]. Puting $Q_1 = H_D/2, Q_2 = (\sigma_2 p - \sigma_1 m)/2$ we obtain

$$Q_1 Q_2 + Q_2 Q_1 = 0 \tag{5.1}$$

$$Q_1^2 = Q_2^2 = \tfrac{1}{2} H \tag{5.2}$$

$$[Q_1, H] = [Q_2, H] = 0 \tag{5.3}$$

where

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \qquad H_\pm = \frac{1}{2}(p^2 + m^2 \pm m') \tag{5.4}$$

is a Hamiltonian of the supersymmetric mechanics.

Due to the algebraic relations (5.1)–(5.3) the point spectra of H_+ and H_- may differ by a single eigenvalue equal to zero only ('zero mode').

The Schrödinger operators $H_\pm$ are Sturm–Liouville differential operators on $L^2[0, a)$ the properties of which are very well known [5, 6]. In the case of Alfvén waves we expect in a generic situation that $a = \xi_\infty < \infty$ (see section 6). Hence the spectrum of $H_\pm$ is discrete in 'all ordinary cases' [6]. Therefore in a generic case the model atmosphere acts like a finite cavity for Alfvén waves.

The situation is different for acoustic waves. Typically, the sound velocity $v_S(z)$ varies slowly with height and may be treated as a constant in the asymptotic region. Then asymptotically, the Dirac equation (4.16) describes a 'free particle'. Therefore the generic spectrum of H_S^2 is continuous with perhaps a finite number of low-lying eigenvalues.

One should remember that in the physical situation the problem of boundary conditions is more complicated. For Alfvén waves a more complete formulation of the wave equations gives a maximum phase velocity of c which modifies reflection and transmission properties for large z. For the acoustic wave nonlinear effects and damping become very important.

6. Hydrostatic isothermal atmosphere: an exactly soluble model

The rigorous spectral analysis of the Dirac Hamiltonians H_A, H_S can be performed for a simple model of a hydrostatic isothermal atmosphere with uniform magnetic and gravity fields [1, 7, 8]. We assume that (4.6) holds and

$$\frac{\partial}{\partial z} g(z) = \frac{\partial}{\partial z} B_z(z) = B_y = B_x = 0 \tag{6.1}$$

$$\frac{p_0}{\rho_0} = \text{constant}. \tag{6.2}$$

Hence

$$\rho_0(z) = \rho_0(0) e^{-z/\Lambda} \qquad p_0(z) = p_0(0) e^{-z/\Lambda} \tag{6.3}$$

$$v_{\mathrm{S}}(z) = c_{\mathrm{S}} = \sqrt{\frac{p_0(0)}{\rho_0(0)}} \tag{6.4}$$

$$\Lambda = \frac{c_{\mathrm{S}}^2}{g} \tag{6.5}$$

$$v_{\mathrm{A}}(z) = c_{\mathrm{A}} \mathrm{e}^{z/2\Lambda} \qquad c_{\mathrm{A}} = \frac{B_z}{\sqrt{\mu\rho_0(0)}}. \tag{6.6}$$

The rescaled height variables are given by

$$\xi = 2\Lambda(1 - \mathrm{e}^{-z/2\Lambda}) \qquad \xi_\infty = 2\Lambda \tag{6.7}$$

$$\eta = z \tag{6.8}$$

and the Dirac Hamiltonians are the following:

$$H_{\mathrm{A}} = c_{\mathrm{A}} \left(-\mathrm{i}\sigma_1 \frac{\partial}{\partial \xi} + \frac{1}{2}\sigma_2 \frac{1}{2\Lambda - \xi} \right) \qquad \xi \in [0, 2\Lambda) \tag{6.9}$$

$$H_{\mathrm{S}} = c_{\mathrm{S}} \left(-\mathrm{i}\sigma_1 \frac{\partial}{\partial z} + \sigma_2 \frac{1}{\Lambda} \right) \qquad z \in [0, \infty). \tag{6.10}$$

The operators (6.9) and (6.10) with fixed boundary conditions possess unique self-adjoint extensions and their (generalized) eigenvalues and eigenfunctions can be exactly calculated. The rigorous results are formulated in the following theorems. We use a standard notation J_l, Y_l for the Bessel functions of the first and the second kind respectively.

Theorem 1. The operator H_{A} formally defined by (6.9) with two different boundary conditions at $\xi = 0$

(*a*) $\psi_1(0) = 0$

(*b*) $\psi_2(0) = 0$

possesses the corresponding two unique self-adjoint extensions H_{A}^a, H_{A}^b characterized by the following spectral properties:

(1*a*)The spectrum of H_{A}^a consists of the non-degenerate eigenvalues $\omega_n, n = 0, \pm 1, \pm 2, \ldots$ given by $\omega_n = \lambda_n c_{\mathrm{A}}/2\Lambda$ where λ_n are all zeros of the function $J_1(\lambda)$ ordered in such a way that $\lambda_0 = 0,\ \lambda_{n+1} > \lambda_n, \lambda_{-n} = -\lambda_n$.

(2*a*)The normalized eigenvectors of H_{A}^a are given by

$$\Psi^n = \begin{pmatrix} \psi_1^n \\ \psi_2^n \end{pmatrix} \qquad \psi_1^0(\xi) = 0 \qquad \psi_2^0(\xi) = \frac{1}{\Lambda}\sqrt{2\Lambda - \xi}$$

$$\psi_1^n(\xi) = \frac{\pi\lambda_n}{2\sqrt{2\Lambda}} Y_1(\lambda_n)\sqrt{1 - \xi/2\Lambda}\, J_1(\lambda_n(1 - \xi/2\Lambda))$$

$$\psi_2^n(\xi) = \mathrm{i}\frac{\pi\lambda_n}{2\sqrt{2\Lambda}}\, Y_1(\lambda_n)\sqrt{1 - \xi/2\Lambda}\, J_0(\lambda_n(1 - (\xi/2\Lambda))$$

(1*b*) The spectrum of H_A^b consists of the non-degenerate eigenvalues ω_m, $m = \pm 1, \pm 2, \ldots$ given by $\omega_m = \lambda_m c_A / 2\Lambda$ where λ_m are all zeros of the function $J_0(\lambda)$ ordered in such a way that $\lambda_{-m} = -\lambda_m$, $\lambda_{m+1} > \lambda_m$.

(2*b*) The normalized eigenvectors of H_A^b are given by

$$\Psi^m = \begin{pmatrix} \psi_1^m \\ \psi_2^m \end{pmatrix}$$

$$\psi_1^m(\xi) = \frac{\pi \lambda_m}{2\sqrt{2\Lambda}} Y_0(\lambda_m) \sqrt{1 - \xi/2\Lambda}\, J_1(\lambda_m(1 - \xi/2\Lambda))$$

$$\psi_2^m(\xi) = \mathrm{i} \frac{\pi \lambda_m}{2\sqrt{2\Lambda}} Y_0(\lambda_m) \sqrt{1 - \xi/2\Lambda}\, J_0(\lambda_m(1 - (\xi/2\Lambda)).$$

Remark. The zero mode ($\omega_0 = 0$) exists for the boundary condition (*a*) only while the case (*b*) corresponds to a 'spontaneous supersymmetry breaking'.

Theorem 2. The operator H_S given formally by (6.10) possesses two unique self-adjoint extensions H_S^a, H_S^b corresponding to the two different boundary conditions

(*a*) $\phi_1(0) = 0$

(*b*) $\phi_2(0) = 0$

and characterized by the following spectral properties:

(1) The spectrum of H_S^a and H_S^b is continuous and consists of all real numbers ω such that $|\omega| \geqslant \Omega = c_S/\Lambda$.

The generalized eigenvectors may be chosen as:

(2*a*) for H_S^a

$$\phi_1^\omega(z) = \frac{1}{\sqrt{\Lambda}} \frac{\omega}{\sqrt{\omega^2 - \Omega^2}} \sin(\sqrt{\omega^2 - \Omega^2}\, z/c_S)$$

$$\phi_2^\omega(z) = \frac{\mathrm{i}}{\sqrt{\Lambda}} [\cos(\sqrt{\omega^2 - \Omega^2}\, z/c_S) - \frac{1}{\Omega} \sqrt{\omega^2 - \Omega^2} \sin(\sqrt{\omega^2 - \Omega^2}\, z/c_S)]$$

(2*b*) for H_S^b

$$\phi_1^\omega(z) = \frac{\mathrm{i}}{\sqrt{\Lambda}} [\cos(\sqrt{\omega^2 - \Omega^2}\, z/c_S) + \frac{1}{\Omega} \sqrt{\omega^2 - \Omega^2} \sin(\sqrt{\omega^2 - \Omega^2}\, z/c_S)]$$

$$\phi_2^\omega(z) = \frac{1}{\sqrt{\Lambda}} \frac{\omega}{\sqrt{\omega^2 - \Omega^2}} \sin\left(\sqrt{\omega^2 - \Omega^2}\, z/c_S\right)$$

(3) the density of states which appears in the eigenfunction expansion

$$H_S = \int_{-\infty}^{\infty} \mathrm{d}\omega\, N(\omega)\, \omega\, |\Phi^\omega >< \Phi^\omega|$$

is given in both cases by

$$N(\omega) = \begin{cases} 0 & \text{for } |\omega| \leqslant \Omega \\ (\pi \Omega \omega)^{-1} \sqrt{\omega^2 - \Omega^2} & \text{for } |\omega| \geqslant \Omega\,. \end{cases}$$

Proof of theorem 1. In order to prove theorem 1 we follow the standard Weyl and Titchmarsh approach to Sturm–Liouville operators which can also be easily adapted to Dirac operators. We refer to the monographs of Richtmyer [5] and Eastham and Kalf [6] and give a brief sketch of the proof only.

We start with the operator (6.9) and to avoid the complicated notation we introduce a dimensionless variable

$$r = 1 - \xi/2\Lambda \qquad r \in (0,1] \tag{6.11}$$

and the 'dimensionless operator'

$$T = \mathrm{i}\sigma_1 \tfrac{\partial}{\partial r} + \sigma_2 \tfrac{1}{2r}. \tag{6.12}$$

The crucial point is to find the solution of the generalized eigenvalue problem

$$T\Psi = \lambda\Psi \tag{6.13}$$

with $\lambda \in C$ and for the specific values of $\psi_1(1), \psi_2(1)$ ($r = 1$ corresponds to $z = 0$).

Elimination of one of the functions, say ψ_2, leads to a second order differential equation for the other, namely

$$\psi_1'' + \left(\lambda^2 - (3/4)\tfrac{1}{r^2}\right)\psi_1 = 0. \tag{6.14}$$

For ψ_2 we have

$$\psi_2 = \frac{\mathrm{i}}{\lambda}\psi_1' + \frac{\mathrm{i}}{2\lambda}\frac{1}{r}\psi_1. \tag{6.15}$$

The solutions of (6.14) and (6.15) are the following

$$\psi_1(r;\lambda) = \sqrt{r}[A(\lambda)J_1(\lambda r) + B(\lambda)Y_1(\lambda r)] \tag{6.16}$$

$$\psi_2(r;\lambda) = \mathrm{i}\sqrt{r}[A(\lambda)J_0(\lambda r) + B(\lambda)Y_0(\lambda r)]. \tag{6.17}$$

We search now for the non-trivial solutions $\psi_j(\cdot;\pm\mathrm{i}) \in L^2(0,1], j = 1,2$. Because $Y_1(\pm\mathrm{i}r)$ is not in $L^2(0,1]$ hence $B(\pm\mathrm{i}) = 0$ in this case. On the other hand $A(\pm\mathrm{i}) \neq 0$ contradicts both boundary conditions

(*a*) $$\psi_1(1;\pm\mathrm{i}) = 0 \tag{6.18}$$

(*b*) $$\psi_2(1;\pm\mathrm{i}) = 0. \tag{6.19}$$

Therefore the deficiency indices of the corresponding symmetric operators T^a, T^b are $(0,0)$ and T^a, T^b are essentially self-adjoint.

We define now two special solutions of (6.12)

$$f_1(r;\lambda) = (\pi\lambda/2)\sqrt{r}[J_1(\lambda)Y_1(\lambda r) - Y_1(\lambda)J_1(\lambda r)] \tag{6.20}$$

$$f_2(r;\lambda) = (\mathrm{i}\pi\lambda/2)\sqrt{r}[J_1(\lambda)Y_0(\lambda r) - Y_1(\lambda)J_0(\lambda r)] \tag{6.21}$$

with the boundary conditions $f_1(1;\lambda) = 0, f_2(1;\lambda) = \mathrm{i}$ and

$$g_1(r;\lambda) = (\pi\lambda/2)\sqrt{r}[Y_0(\lambda)J_1(\lambda r) - J_0(\lambda)Y_1(\lambda r)] \tag{6.22}$$

$$g_2(r;\lambda) = (\mathrm{i}\pi\lambda/2)\sqrt{r}[Y_0(\lambda)J_0(\lambda r) - J_0(\lambda)Y_0(\lambda r)] \tag{6.23}$$

with the boundary conditions $g_1(1;\lambda) = 1, g_2(1;\lambda) = 0$. We have used the relation $J_1(x)Y_0(x) - Y_1(x)J_0(x) = 2/\pi x$.

According to the general theory the spectra of T^a and T^b (we use the same symbols for the unique self-adjoint extensions) are determined by the properties of the functions $m_a(\lambda), m_b(\lambda)$ defined as

$$m_a(\lambda) = \lim_{r\to 0} \frac{g_j(r;\lambda)}{f_j(r;\lambda)} = \frac{J_0(\lambda)}{J_1(\lambda)} \tag{6.24}$$

$$m_b(\lambda) = \lim_{r\to 0} \frac{f_j(r;\lambda)}{g_j(r;\lambda)} = -\frac{J_1(\lambda)}{J_0(\lambda)} \tag{6.25}$$

$$\operatorname{Im}\lambda \neq 0 \qquad j = 1, 2.$$

Namely the function

$$\rho(\lambda) = \frac{1}{2\pi i} \int_{0+}^{\lambda} [m(\mu + i\epsilon) - m(\mu - i\epsilon)]\, d\mu \tag{6.26}$$

appears in the eigenfunction expansion of the operator and formally $d\rho(\lambda)/d\lambda$ is a 'density of states'. In our case $m_a(\lambda), m_b(\lambda)$ are meromorphic functions with simple poles at the zeros λ_n^a or λ_m^b of the functions J_1, J_0 respectively. Hence the functions ρ_a, ρ_b are almost everywhere constant with jumps (all equal to 1 with the exception of the jump equal to 2 at $\lambda_0^a = 0$ for ρ_a) at the eigenvalues λ_n^a, λ_m^b respectively. The normalized eigenfunctions for T^a, T^b are given by ((6.20) and (6.21)) or ((6.22) and (6.23)) with proper values of λ_n^a, λ_m^b respectively (one should remember factor $\sqrt{2}$ for λ_0^a).

The statements of theorem 1. are obtained by the proper rescaling of the variable $r \mapsto \xi$ and the operator $T \mapsto H_A$. □

Proof of theorem 2.. We consider a dimensionless operator

$$D = -i\sigma_1 \frac{\partial}{\partial s} + \sigma_2 \qquad s \in [0, \infty) \tag{6.27}$$

on $L^2[0,\infty) \otimes C^2$ with the simple solution of the generalized eigenvalue problem

$$\phi_1(s;\lambda) = A(\lambda)e^{i\sqrt{\lambda^2-1}s} + B(\lambda)e^{-i\sqrt{\lambda^2-1}s} \tag{6.28}$$

$$\phi_2(s;\lambda) = -\lambda^{-1}A(\lambda)[\sqrt{\lambda^2-1} + i]e^{i\sqrt{\lambda^2-1}s} + \lambda^{-1} B(\lambda)[\sqrt{\lambda^2-1} - i]e^{-i\sqrt{\lambda^2-1}s}. \tag{6.29}$$

Following the same pattern as in the proof of the theorem 1 we obtain the continuous 'density of states' in the both cases (a) and (b)

$$\frac{d\rho}{d\lambda} = \begin{cases} (\pi\lambda)^{-1}\sqrt{\lambda^2-1} & \text{for } \lambda^2 > 1 \\ 0 & \text{for } \lambda^2 < 1 \end{cases}$$

and after rescaling the statements of theorem 2 are proved. □

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